

Minimal Model Program

Learning Seminar.

Week 16 :

- Asymptotic multiplier ideals.

Multiplier ideals:

X an algebraic variety and $D \geq 0$ a divisor on X .

The multiplier ideal $\mathcal{J}(X, c \cdot D) \subseteq \mathcal{O}_X$ measures the singularities of the pair (X, D) . supported on the "most sing loc." of the pair.

~~smooth~~ ~~XX~~

- (X, cD) log smooth and $\text{coeff}(cD) < 1$, then $\mathcal{J}(X, cD) = \mathcal{O}_X$.

- $X = \mathbb{A}_{x,y}^2$, $C = \text{div}(xy)$, $\mathcal{J}(X, D) = \langle xy \rangle$. +

- $X = \mathbb{A}_{x,y}^2$, $C = \text{div}(x^2 - y^3)$, $\mathcal{J}(X, c \cdot D) = \mathcal{O}_X$ for $c < \frac{5}{6}$.

$$\mathcal{J}(X, \frac{5}{6}D) = \langle x, y \rangle.$$

Philosophy: $\mu: X' \longrightarrow X$ log resolution of (X, D) .

$$\mu^*(K_X + D) = K_{X'} + D' \quad \text{may not be effective.}$$

$$F = \left\lceil \sum_{\substack{P \text{ prime on } D' \\ \text{coeff}_P(D') \geq 0}} \text{coeff}_P(D') \right\rceil \quad \begin{array}{l} \text{coeff } D' \leq 1 \\ < 1 \\ \text{sing are good.} \end{array}$$

$$\text{Then } \mathcal{J}(X, D) = \mu_* \mathcal{O}_{X'}(-F).$$

i.e., $\mathcal{J}(X, D)$ is the ideal measuring the failure of (X, D) from being log canonical.

Multiplication ideals:

Theorem (Nadel vanishing): Let X be a smooth proj variety, $D \geq 0$ \mathbb{Q} -divisor on X . L Cartier divisor on X such that

$L - cD$ is big & nef, $c > 0$.

Then, $H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(cD)) = 0$.

Theorem (Restriction): X smooth variety, $D \geq 0$ \mathbb{Q} -divisor, $H \subseteq X$ smooth hypersurface not contained in $\text{supp}(D)$. Then, there is an inclusion:

$$\mathcal{J}(H, D_H) \subseteq \mathcal{J}(X, D) \cdot \mathcal{O}_H.$$

Theorem (Skoda's Theorem): X smooth variety of dimension n , $\alpha \subseteq \mathcal{O}_X$ ideal sheaf on X . If $c \geq n$, then

$$\mathcal{J}(\alpha^c) = \alpha \cdot \mathcal{J}(\alpha^{c-n}).$$

In particular, if $m \geq n$, then $\mathcal{J}(\alpha^m) \subseteq \alpha^{m-n+1}$.

Asymptotic multiplier ideals:

\mathcal{L} linear system on X .

$A_1, \dots, A_k \in |\mathcal{L}|$ general $k > c$.

$$D = \frac{A_1 + \dots + A_k}{k}$$

$$\mathcal{J}(X, c \cdot \mathcal{L}) = \mathcal{J}(X, c \cdot D).$$

$D \in |\mathcal{L}^m|$ for every possible $m \in \mathbb{Z}_{>0}$.

$$\mathcal{J}(X, \frac{D}{m})$$

Notation: $N(L) = N(X, L) = \{ m \geq 0 \mid H^0(X, \mathcal{O}_X(mL)) \neq 0 \}$.

$e(L)$ is the exponent of this monoid. (g.c.d of elements in $N(L)$).

$L \leadsto e(L)L$, assume $e(L) = 1$.

$m \geq 0, H^0(X, \mathcal{O}_X(mL)) \neq 0$.

$\text{char } k = 0, \bar{k} = k$

Lemma: X a smooth variety over k .

L a divisor on X . Then, for any $k \geq l$, we have

$$\mathcal{J}\left(\frac{c}{p} \cdot \underline{|pL|}\right) \subseteq \mathcal{J}\left(\frac{c}{pk} \cdot \underline{|pkL|}\right)$$

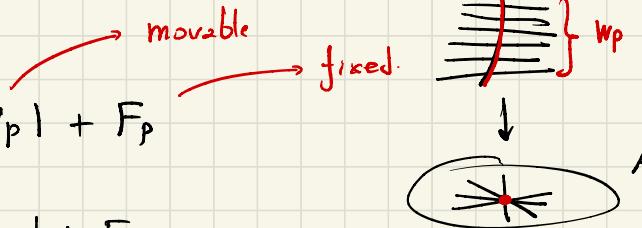
less singular.

Proof: Assume $|pL| \neq \emptyset$. $\mu: X' \rightarrow X$ log resolution

of $|pL|$ and $|pkL|$.

$$\mu^*(|pL|) = |W_p| + F_p$$

$$\mu^*(|pkL|) = |W_{pk}| + F_{pk}$$



We have a natural map $S^k W_p \rightarrow W_{pk}$

whose image is a free linear subsystem of $\mu^*(|pkL|)$.

and its fixed divisor is kF_p . Hence.

$$kF_p \geq F_{pk}$$

$$KF_p \Rightarrow F_{pk}.$$

$$\begin{aligned} J\left(\frac{c}{p} |pL|\right) &= \mu_* \mathcal{O}_X^* (K_{X'/X} - \left[\frac{c}{p} F_p\right]) \\ &\subseteq \mu_* \mathcal{O}_X^* (K_{X'/X} - \left[\frac{c}{pk} F_{pk}\right]) \\ &= J\left(\frac{c}{pk} |pKL|\right). \end{aligned}$$

□

$$J\left(\frac{c}{p} |pL|\right) \subseteq J\left(\frac{c}{2p} |2pL|\right) \subseteq \dots$$

By ACC of ideals there is a maximal element.

$$J\left(\frac{c}{p} |pL|\right) = J\left(\frac{c}{q} |qL|\right)$$

$$\cap \quad \cap$$

$$J\left(\frac{c}{pq} |pqL|\right)$$

If p is large & divisible enough $J\left(\frac{c}{p} |pL|\right)$ is a unique maximal ideal of the set.

If p is large & divisible enough $\mathcal{J}\left(\frac{c}{p} \cdot |pL|\right)$ is
 a unique maximal ideal of the set.

$$\left\{ \mathcal{J}\left(\frac{c}{k} \cdot |kL|\right) \mid k > 0 \right\}.$$

Definition: X smooth projective, L is a divisor with non-negative Iitaka dimension. Then, we define

$$\mathcal{J}(c \parallel L \parallel) = \mathcal{J}(X, c \cdot \parallel L \parallel) \subseteq \mathcal{O}_X$$

is defined to be the unique maximal member among the family of ideals

$$\left\{ \mathcal{J}\left(\frac{c}{p} \cdot |pL|\right) \right\}.$$

L is a f.g. divisor.

Example: $R(X, L) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mL))$. f.g.

There exists $s \geq 1$ so that for $m \gg 0$ all sections of $H^0(X, \mathcal{O}_X(mL))$ can be written as 1.c of products of sections of $H^0(X, \mathcal{O}_X(kL))$ for $1 \leq k \leq s$.

Example: L semistable $\implies L$ is a f.g. division

Remark: Assume L is f.g. Then for some $r \geq 0$, we have

$$\mathcal{J}(X, c \cdot \text{rk } L) = \mathcal{J}(X, c \cdot \|\text{rk } L\|)$$

for every $c \geq 0$.

Remark: Even if the sections of $|mL|$ do not stabilize, then the singularities of those sections will stabilize.

Example: If D semistable, then $\mathcal{J}(\|mD\|) = \mathcal{O}_X$.

Thm (elementary properties): L integral with non-negligible Iitaka dimension, $c > 0$ fixed.

i) $\mathcal{J}(c \cdot \|mL\|) = \mathcal{J}(mc \cdot \|L\|)$

ii) $\mathcal{J}(c \cdot \|mL\|) \supseteq \mathcal{J}(c \cdot \| (m+1)L \|)$

iii) $b_m = b(|mL|) \subseteq \mathcal{O}_X$, then $b_m \cdot \mathcal{J}(\|mL\|) \subseteq \mathcal{J}(\|(m+1)L\|)$

iv) $b_m \subseteq \mathcal{J}(\|mL\|)$ for all $m \geq 1$

$$iii) \quad b_m = b(\text{Im } L) \subseteq \mathcal{O}_x, \quad \text{then} \quad \underline{b_m \cdot J(\|eL\|)} \subseteq \underline{J(\|cm+e\|L\|)}$$

Proof of iii): $\text{Im } L \neq \emptyset$, $p \gg 0$ and $\mu: X' \rightarrow X$

common resolution of

$$\|p\ell L\|, \quad \|mL\|, \quad \|pmL\|, \quad \|p(cm+e)L\|.$$

$\left. \begin{matrix} \\ \end{matrix} \right\}$ fixed part

$$F_{pl}, \quad F_m, \quad F_{pm}, \quad F_{pcm+e}.$$

There are the following inequalities:

$$\underbrace{pF_m + F_{pl}}_{\text{fixed part}} \geq F_{pm} + F_{pl} \geq \underbrace{F_{pcm+e}}_{\text{fixed part}}$$

for $p \gg 0$ and
divisible energy

$$\text{Consequently,} \quad -F_m - \left[\frac{1}{p} F_{pl} \right] \leq - \left[\frac{1}{p} F_{pcm+e} \right].$$

$$b_m \cdot J(\|eL\|) = \mu_* (\mathcal{O}_{X'}(-F_m)) \cdot \mu_* (\mathcal{O}_{X'}(K_{X'/X} - \left[\frac{1}{p} F_{pl} \right]))$$

$$= \mu_* (\mathcal{O}_{X'}(K_{X'/X} - F_m - \left[\frac{1}{p} F_{pl} \right]))$$

$$\subseteq \mu_* (\mathcal{O}_{X'}(K_{X'/X} - \left[\frac{1}{p} F_{pcm+e} \right]))$$

$$= J(\|(cm+e)L\|)$$

□

Graded system of ideals:

$$\alpha_0 = \{\alpha_m\}_{m \geq 0}, \text{ so that } \alpha_k \subseteq \mathcal{O}_X.$$

$$\alpha_0 = \mathcal{O}_X, \quad \alpha_m \cdot \alpha_l \subseteq \alpha_{m+l} \text{ for all } m, l \geq 0.$$

Lemma: α_0 is a graded system of ideals on X . Then

$$\mathcal{J}\left(\frac{c}{p} \cdot \alpha_p\right) \subseteq \mathcal{J}\left(\frac{c}{p^k} \cdot \alpha_{pk}\right).$$

for all integers $p, k \geq 1$

Definition: $\mathcal{J}(c \cdot \alpha_0) = \text{unique maximal element of}$
 $\{\mathcal{J}\left(\frac{c}{p} \cdot \alpha_p\right)\}.$

Theorem (Formal properties): $\alpha_0 = \{\alpha_m\}$ on X ,

and fix $c > 0$. Then, we have the following:

i) $\mathcal{J}(cm \cdot \alpha_0) = \mathcal{J}\left(\frac{c}{p} \cdot \alpha_{pm}\right)$ for p large & divisible.

ii) $\mathcal{J}(c \cdot \alpha_0) \supseteq \mathcal{J}(d \cdot \alpha_0)$ for $d \geq c$.

iii) $\alpha_m \cdot \mathcal{J}(\alpha_0^k) \subseteq \mathcal{J}(\alpha_0^{m+k})$.

iv) for every $m \geq 1$, $\alpha_m \subseteq \mathcal{J}(\alpha_0^m)$

Properties of asymptotic multiplier ideals:

$Y \subseteq X$, α graded system of ideals on X

$\alpha_{\cdot, Y} = \alpha \cdot \mathcal{O}_Y$ graded system of ideals
on Y by restriction.

Theorem: Let $Y \subseteq X$ smooth subvariety.

α graded system of ideals on X . Then

$$\mathcal{J}(Y, c \cdot \alpha_{\cdot, Y}) \subseteq \mathcal{J}(X, c \cdot \alpha)_{\cdot, Y}$$

for every real $c > 0$.

Proof: Fix $p >> 0$ computing both asympt mult ideals.

$$\begin{aligned} \mathcal{J}(Y, c \cdot \alpha_{\cdot, Y}) &= \mathcal{J}(Y, \frac{c}{p} \cdot \alpha_{p, Y}) \\ &\subseteq \mathcal{J}(X, \frac{c}{p} \cdot \alpha_p)_{\cdot, Y} \\ &= \mathcal{J}(X, c \cdot \alpha)_{\cdot, Y} \end{aligned}$$

restriction thm
of multiplier
ideals

□

Theorem (generic restriction): $f: X \rightarrow T$ surjective smooth map between smooth varieties. $\alpha_0 = \{\alpha_k\}$ system of ideals on X .

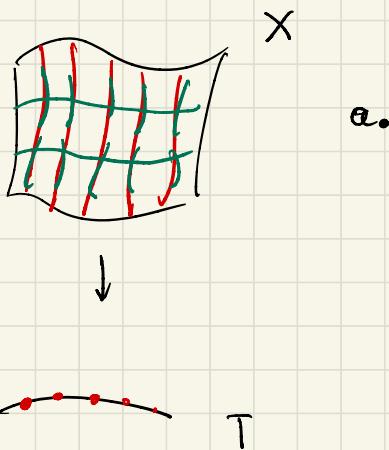
$\alpha_{0,t}$ the system of ideals on X_t .

$B \subsetneq T$ countable union of closed proper subsets, such that if $t \in T - B$, then

$$\mathcal{J}(X_t, c \cdot \alpha_{0,t}) = \mathcal{J}(X, c \cdot \alpha_0)_t.$$

for every $c > 0$.

Pictorially:



Proof: There exists $B_p \subsetneq T$ such that

$$J(X, d \cdot \alpha_p)_t = J(X_t, d \cdot \alpha_{p,t})$$

for all $t \in T - B_p$ and all $d > 0$.

$$B = \bigcup_{p \geq 0} B_p.$$

and fix $c > 0$.

There exists $p \gg 0$ (depending on c) such that

$$J(X, c \cdot \alpha_p) = J(X, \frac{c}{p} \cdot \alpha_p)$$

If $t \in T - B$, by construction

$$J(X, \frac{c}{p} \cdot \alpha_p) = J(X_t, c \cdot \alpha_{p,t}), \text{ so}$$

\supseteq

$$J(X, c \cdot \alpha_p)_t = J(X_t, \frac{c}{p} \cdot \alpha_{p,t}) \subseteq J(X_t, c \cdot \alpha_{p,t})$$



mix element-

The inclusion \supseteq is provided by restriction Theorem

D.

Theorem (Nadel vanishing): X smooth projective variety,
 L an integral divisor on X . (of non-neg Iitaka dimension).

A ~~nef~~ & big integral divisor on X . Then

$$H^i(X, \mathcal{O}_X(K_X + mL + A) \otimes J(\|mL\|)) = 0$$

for all $i > 0$.

can be dropped.

Theorem (Kollar): X smooth projective with K_X nef

$Y \xrightarrow{f} X$ finite étale & big.

Then $P_m(Y) = f_* P_m(X)$.

$$\parallel \qquad \qquad \qquad \parallel$$

$$h^0(Y, \mathcal{O}_Y(mK_Y)) \qquad h^0(X, \mathcal{O}_X(mK_X)).$$

$$\text{Proof: } H^0(X, \mathcal{O}_X(mK_X))$$

||

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(\|mK_X\|))$$

||

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(\|(m-1)K_X\|)).$$

By Nadel vanishing  has vanishing higher H^i 's.

$$H^i(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(\|(m-1)K_X\|)) = 0$$

for $i > 0$.

$$h^0(X, \mathcal{O}_X(mK_X)) = \chi(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(\|(m-1)K_X\|))$$

$$h^0(Y, \mathcal{O}_Y(mK_Y)) = \chi(Y, \mathcal{O}_Y(mK_Y) \otimes \mathcal{J}(\|(m-1)K_Y\|))$$

Since f is étale.

$$\mathcal{O}_Y(mK_Y) \otimes \mathcal{J}(\|(m-1)K_Y\|) = f^*(\mathcal{O}_X(mK_X) \otimes \mathcal{J}(\|(m-1)K_X\|))$$

$$\chi(Y, \mathcal{O}_Y(mK_Y)) = \int \chi(X, \mathcal{O}_X(mK_X))$$

||

$$h^0(Y, \mathcal{O}_Y(mK_Y)) = \int h^0(X, \mathcal{O}_X(mK_X)). \quad \square$$